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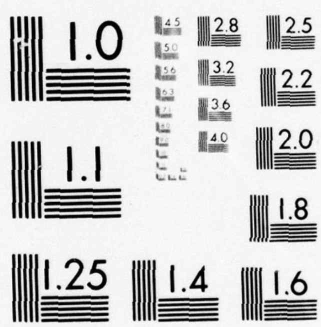
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BIFURCATION NEAR FAMILIES OF SOLUTIONS+

by

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+This research was supported in part by the Air Force Office of Scientific Research under AFOSR 71-2078, in part by the National Science Foundation under MPS 71-02923 and in part by the U.S. Army Research Office under AROD AAG 29-76-6-0052.

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BIFURCATION NEAR FAMILIES OF SOLUTIONS

Jack K. Hale

Summary: Many investigations in bifurcation theory are concerned with the following problem. If $M(0,0) = 0$ and $\partial M(0,0)/\partial x$ has a nontrivial null space, find all solutions of the equation

$$M(x,\lambda) = 0 \tag{1.1}$$

for (x,λ) in a neighborhood of $(0,0) \in X \times \Lambda$.

If $\dim \Lambda = 1$; that is, there is only one parameter involved then the existence of more than one solution in a neighborhood of zero can be proved by making assumptions only about $\partial M(0,0)/\partial x$ and $\partial M(0,0)/\partial x \partial \lambda$. However, if $\dim \Lambda \geq 2$, then the problem is much more difficult and more detailed information is needed about the function M. A careful examination of the existing literature for $\dim \Lambda \geq 2$ reveals that the additional conditions imposed on M imply, in particular, that the solution $x = 0$ of the equation

$$M(x,0) = 0 \tag{1.2}$$

is isolated (see, for example, the papers on catastrophe theory). These hypotheses eliminate the possibility that Equation (1.2) has a family of solutions containing $x = 0$. Such a situation occurs, for example, for $M(x,\lambda) = Ax + N(x,\lambda)$, where A is linear with a nontrivial null space and $N(x,0) = 0$ for all x. There also are interesting applications where Equation (1.2) is nonlinear and there exists a family of solutions. For example, Equation (1.2) could be an autonomous ordinary differential equation with a nonconstant periodic orbit of period 2Π with the family of solutions being

Summary (continued)

obtained by a phase shift. When the differential equation in the latter situation is a Hamiltonian system, the parameters (λ_1, λ_2) could correspond to a small damping term and a small forcing term of period 2Π. To the author's knowledge, the first complete investigations of special problems of each of these latter types are contained in papers by Hale, Táboas and Rodrigues.

It is the purpose of this paper to begin the investigation of the abstract problem for Equation (1.1), especially to extend the results in the paper by Hale and Táboas.

BIFURCATION NEAR FAMILIES OF SOLUTIONS

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§1. Introduction and statement of problem.

Suppose X, Z, Λ are Banach spaces, $M: X \times \Lambda \to Z$ is continuous together with its Fréchet derivatives up through order two. Many investigations in bifurcation theory are concerned with the following problem. If $M(0,0) = 0$ and $\partial M(0,0)/\partial x$ has a nontrivial null space, find all solutions of the equation

$$M(x,\lambda) = 0 \tag{1.1}$$

for (x,λ) in a neighborhood of $(0,0) \in X \times \Lambda$.

If $\dim \Lambda = 1$; that is, there is only one parameter involved, then the existence of more than one solution in a neighborhood of zero can be proved by making assumptions only about $\partial M(0,0)/\partial x$ and $\partial M(0,0)/\partial x \partial \lambda$. However, if $\dim \Lambda \geq 2$, then the problem is much more difficult and more detailed information is needed about the function M. A careful examination of the existing literature for $\dim \Lambda \geq 2$ reveals that the additional conditions imposed on M imply, in particular, that the solution $x = 0$ of the equation

$$M(x,0) = 0 \tag{1.2}$$

is isolated (see, for example, [1], [2] and the papers on catastrophe theory in [3]). These hypotheses eliminate the possibility that Equation (1.2) has a family of solutions containing $x = 0$. Such a situation occurs, for example, for $M(x,\lambda) = Ax + N(x,\lambda)$, where A is linear with a nontrivial null space and $N(x,0) = 0$ for all x. There also are interesting applications where Equation (1.2) is nonlinear and there exists a family of solutions. For example, Equation (1.2) could be an autonomous ordinary differential equation with a nonconstant periodic orbit of period 2Π with the family of solutions being obtained by a phase shift. When the differential equation in the latter situation is a Hamiltonian system, the parameters (λ_1, λ_2) could correspond to a small damping term and a small forcing term of period 2Π. To the author's knowledge, the first complete investigations of special problems of each of these latter types are contained in [4], [5].

It is the purpose of this paper to begin the investigation of the abstract problem for Equation (1.1), especially to extend the results in [5]. More specifically, suppose Equation (1.2) has a one parameter family of solutions $x = p(t)$, $0 \leq t \leq 1$, $p(0) = p(1)$, which is continuous together with derivatives up through order two with $p'(t) = dp(t)/dt \neq 0$, $p'(0) = p'(1)$, $p''(0) = p'(1)$. Boundary conditions on p are imposed only to avoid a special discussion at the end points of the curve defined by p. Since $M(p(t),0) = 0$ for all t, it follows that $p'(t)$ is a nonzero element of the null space $\mathcal{N}(\Lambda(t))$ of the linear operator

$$\Lambda(t) = \partial M(p(t),0)/\partial x \tag{1.3}$$

for $0 \leq t \leq 1$.

If $\Gamma = \{p(t),\ 0 \leq t \leq 1\} \subseteq X$, the problem is to characterize the solutions of Equation (1.1) in a neighborhood of $\Gamma \times \{0\} \subseteq X \times \Lambda$. Suppose $\mathscr{R}(\Lambda(t))$ is the range of $\Lambda(t)$. For the case in which $\dim \mathcal{N}(\Lambda(t)) = 1 = \operatorname{codim} \mathscr{R}(\Lambda(t))$, and $\Lambda = \mathbb{R}^2$, we give a solution to this problem under certain hypotheses on $\partial M(p(t),0)/\partial\lambda$. One important implication of the results can easily be stated. Suppose $\gamma \subseteq \Lambda = \mathbb{R}^2$ is a continuous curve, $0 \notin \gamma$, $0 \in \mathrm{Cl}\,\gamma$, the closure of γ, and suppose $x(\lambda)$ is a solution of system (1.1) defined and continuous for $\lambda \in \gamma$. If the set $x(\gamma) \subseteq X$ remains in a sufficiently small neighborhood of Γ and the set $x(\gamma)$ is precompact, then all limit points of $x(\lambda)$ as $\lambda \in \gamma$ approaches zero belong to Γ, but $x(\lambda)$ has a limit as $\lambda \to 0$ if and only if $\cot^{-1}(\lambda_1/\lambda_2)$ approaches a constant as $\lambda \in \gamma$ approaches zero.

§2. Statement and implications of results.

For any Banach spaces X, Z, we let $C^k(X,Z)$ be the linear space of all functions from X to Z which are continuous together with all derivatives up through order k. If no confusion may arise, we sometimes write C^k for $C^k(X,Z)$. For any finite collection of elements $q_1,\dots,q_k$ of a Banach space, we let $[q_1,\dots,q_k]$ denote the linear subspace spanned by $q_1,\dots,q_k$. By our boundary condition on p, we may suppose $p \in C^2(\mathbb{R},X)$ and is 1-periodic, that is, periodic of period 1. Suppose $[p'(t)] = \mathcal{N}(\Lambda(t))$ and there is a $q \in C^2(\mathbb{R},Z)$ 1-periodic, such that $[q(t)] \oplus \mathscr{R}(\Lambda(t)) = Z$. If $\mathscr{B}(X)$ is the Banach space of bounded linear operators on X, let $U \in C^2(\mathbb{R}, \mathscr{B}(X))$ be such that $U(t)$ is a projection onto $\mathcal{N}(\Lambda(t))$ and let $E \in C^2(\mathbb{R}, \mathscr{B}(Z))$, $E(t)$ a projection onto $\mathscr{R}(\Lambda(t))$, $I - E(t)$ a projection onto $[q(t)]$. Also suppose U, E are 1-periodic.

If $\Lambda = \mathbb{R}^2$, $\lambda = (\lambda_1,\lambda_2) \in \Lambda$, define $\alpha_j \in C^2(\mathbb{R},\mathbb{R})$, $j = 1,2$, 1-periodic, by the relation

$$\alpha_j(t)q(t) = (I-E(t))\partial M(p(t),0)/\partial\lambda_j \tag{2.1}$$

Our first hypothesis on $\alpha(t) = (\alpha_1(t),\alpha_2(t))$ is

(H_1) $\qquad \alpha(t) \neq 0$ for $t \in \mathbb{R}$.

If $\beta(t) = (\alpha_2(t),-\alpha_1(t))$, then $\beta(t) \neq 0$ by Hypothesis (H_1) and we can let $\phi(t)$ be the angle measured in the counterclockwise direction which $\beta(t)$ makes with the horizontal axis. The function $\phi \in C^2(\mathbb{R},\mathbb{R})$ and is 1-periodic. We impose the following hypotheses on ϕ:

(H_2) The function $\phi'(t)$ has at most a finite set of zeros $\{t^k, k = 1,2,\dots,n\} \subseteq [0,1)$ and $\phi''(t^k) \neq 0$ for $k = 1,2,\dots,n$.

(H_3) $\phi(t^j) \neq \phi(t^k)$, $j \neq k$, $j,k = 1,2,\dots,n$.

We now state the main results of the paper together with implications. The proofs will be given in Section 3. Suppose γ is a smooth curve in $\mathbb{R}^2$ through the origin. If for any $q \in \gamma$, $q \neq (0,0)$, L_q^1 denotes a positively oriented normal to L_q at q, we say γ is crossed from right to left at q if γ is crossed by moving along L_q^1 in the positive direction.

Theorem 2.1. If Hypotheses (H_1)-(H_3) are satisfied, then there exist neighborhoods U of Γ, V of $\lambda = (0,0)$, and $s_0 > 0$, such that, for each $t^j \in \{t^k, k = 1,2,\dots,n\}$, there corresponds a unique curve $\mathscr{L}_j \subseteq V$, tangent to the line $\alpha(t^j)\cdot\lambda = 0$ at zero, $\mathscr{L}_j \cap \partial V \neq \phi$, each $\mathscr{L}_j$ intersects lines through the origin in at most one nonzero point, these curves intersect only at $(0,0)$, the number of solutions of Equation (1.1) increases (or decreases) by exactly two as $\mathscr{L}_j$ is crossed from right to left if t^j is a relative minimum (or maximum) of ϕ.

The curves $\mathscr{L}_j$ can be defined parametrically in the form $\lambda = s\beta_j(s)$, $0 \leq s < s_0$ where $\beta_j \in C^2([0,s_0),\mathbb{R}^2)$, $|\beta_j(s)| = 1$, $0 \leq s < s_0$, and $\alpha(t^j)\cdot\beta_j(0) = 0$. If t_*,t^* are the absolute minimum, maximum, respectively, of ϕ and

$$S(V) = \{\lambda \in V: \lambda\cdot\beta^*(s) < 0 < \lambda\cdot\beta_*(s),\ 0 \leq s < s_0\},$$

where $\lambda = s\beta^*(s)$, $\lambda = s\beta_*(s)$ are the curves corresponding to t^* and t_*, then there are no solutions of Equation (1.1) in U for $\lambda \in S(V)$, at least two in $S^c(V) = V\setminus S(V)$ and all solutions are distinct in the interior of $S^c(V)$.

The curves $\mathscr{L}_j$ in Theorem 2.1 are called the <u>bifurcation curves</u>. To see how easy it is to obtain the complete qualitative picture of the bifurcations near $\lambda = 0$, let us consider a few special cases. If $\phi(t)$ has only one maximum at t^* and one minimum at t_*, there are only two bifurcation curves $\mathscr{L}_*, \mathscr{L}^*$ corresponding to t_*, t^*, respectively. There are no solutions of Equation (1.1) in U for $\lambda \in S(V)$ and exactly two solutions in $S^c(V)$ which are distinct in the interior of $S^c(V)$ (see Figure 1). If there are two maxima and two minima (there must always be an even number of maxima and minima by periodicity), then the situation is depicted in Figure 2. By changing the function ϕ, one can obtain every possible rotation of these pictures.

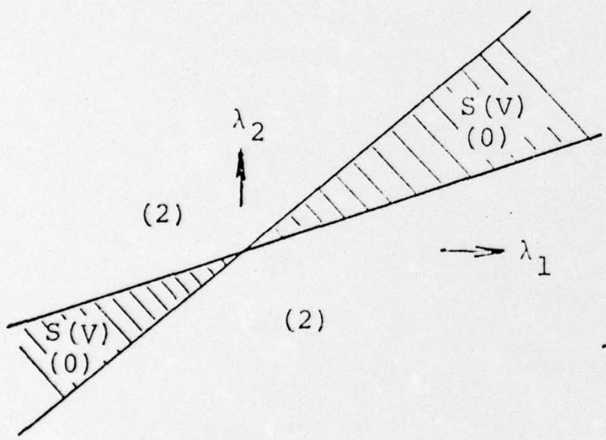

Figure 1.

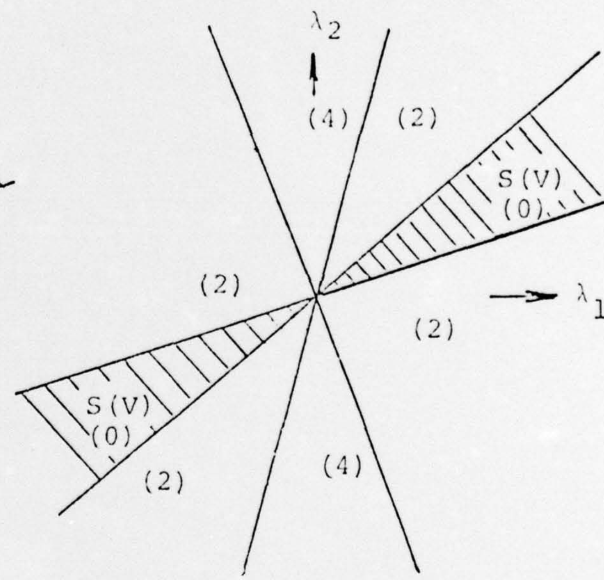

Figure 2.

Another interesting special case is $\alpha_1(t) = -1$, $t \in \mathbb{R}$. Hypothesis (H_1) is always satisfied, $\beta(t) = (\alpha_2(t), 1)$, $\phi(t) = \cot^{-1}\alpha_2(t)$ and the hypotheses (H_1), (H_2) are equivalent to

(H_2') The function α_2' has a most finite set of zeros $\{t^k, k = 1,2,\ldots,n\} \subseteq [0,1)$ and $\alpha_2''(t^k) \neq 0$, $k = 1,2,\ldots,n$.

(H_3') $\alpha_2(t^j) \neq \alpha_2(t^k)$, $j \neq k$, $j, k = 1,2,\ldots,n$.

Theorem 2.1 for this case is essentially contained in [5]. Since $\alpha_1(t) = -1$, the set $S(V)$ must contain the λ_1-axis and, thus, the bifurcation diagram is a rotated version of the ones in Figures 1 and 2.

The following result gives some information about the possible behavior of the solutions of Equation (1.1) as $\lambda \to 0$.

Theorem 2.2. Suppose Hypotheses (H_1)-(H_3) are satisfied, U,V are the neighborhoods given in Theorem 2.1, and suppose γ is a continuous curve defined parametrically by $\lambda_1 = \lambda_1(\tau)$, $\lambda_2 = \lambda_2(\tau)$, $0 \le \tau \le 1$, and $\lambda_1^2(\tau) + \lambda_2^2(\tau) = 0$ if and only if $\tau = 0$. Also, suppose $\gamma \subseteq V$ and for each point $(\lambda_1(\tau), \lambda_2(\tau)) \in \gamma$, there is a solution $x(\tau) \in U$ of Equation (1.1) which is continuous in τ on the half open interval $(0,1]$. If

$$\mathcal{S}(\gamma) = \{x(\tau),\ \ 0 < \tau \le 1\} \subseteq X \tag{2.2}$$

is precompact, and

$$\begin{aligned} \phi_m(\gamma) &= \lim\inf_{\tau\to 0} \cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau)) \\ \phi_M(\gamma) &= \lim\sup_{\tau\to 0} \cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau)) \end{aligned} \tag{2.3}$$

then there is an interval $I(\gamma) \subseteq [0,1]$ such that $\phi(I(\gamma)) = [\phi_m(\gamma), \phi_M(\gamma)]$ and

$$(\mathrm{Cl}\, \mathcal{S}(\gamma)) \backslash \mathcal{S}(\gamma) = \{p(t),\ \ t \in I(\gamma)\}. \tag{2.4}$$

A consequence of the above result is the following

Corollary 2.1. If $\gamma, x(\tau)$ satisfy the conditions of Theorem 2.2, then a necessary and sufficient condition that $x(\tau)$ have a limit as $\tau \to 0$ is that $\cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau))$ has a limit ϕ_0 as $\tau \to 0$. In this case, $x(\tau) \to p(t_0)$ where $t_0 \in [0,1)$ is a solution of the equation $\cot^{-1}(\lambda_1(t)/\lambda_2(t)) = \phi_0$.

The fact that one can obtain solutions which are not continuous in λ at $\lambda = 0$ is not surprising. Consider the scalar equation $\lambda_1 x - \lambda_2 = 0$ which has the solution $x = \lambda_2/\lambda_1$ for $\lambda_1 \ne 0$. Along a curve $\gamma \in \mathbb{R}^2$, this solution has a limit as $\lambda \to 0$ in γ if and only if λ_1/λ_2 approaches a limit as $\lambda \to 0$ in γ.

Let us now make a more interesting application to the second order scalar ordinary differential equation

$$\frac{d^2x}{ds^2} + g(x) + \lambda_1 h(s)\frac{dx}{ds} - \lambda_2 f(s) = 0 \tag{2.5}$$

where h, f are continuous and 1-periodic, $g \in C^2(\mathbb{R},\mathbb{R})$, $xg(x) > 0$ for $x \ne 0$. For $\lambda_1 = \lambda_2 = 0$, the equation

$$\frac{d^2x}{ds^2} + g(x) = 0 \tag{2.6}$$

has a general solution of the form $x = \psi(\omega(a)s + t, a)$, $(a,t) \in \mathbb{R}^2$, where $\psi(\zeta,a) = \psi(\zeta+1,a)$ for all (ζ,a), and $(a,0) = (x(0), dx(0)/ds)$. We suppose Equation (2.6) has a nondegenerate 1-periodic orbit; that is,

There is an $a_0 > 0$ such that $\omega(a_0) = 1$, $d\omega(a_0)/da \ne 0$. (2.7)

Let $Z = \{y: \mathbb{R} \to \mathbb{R}$ which are continuous and 1-periodic$\}$ and use the supremum norm on Z. Let $X = \{y \in Z: y$ has continuous derivatives up through order two$\}$ and use the usual C^2 norm on X. If we define $M: X \times \Lambda \to Z$, $\Lambda = \mathbb{R}^2$ by

$$M(x,\lambda)(s) = \frac{d^2x(s)}{ds^2} + g(x(s)) + \lambda_1 h(s)\frac{dx(s)}{ds} - \lambda_2 f(s)$$

then we are in a position to apply the previous results. In fact, if $p(t)(s) = \psi(\omega(a_0)s + t, a_0)$, then $p(t) \in X$ and satisfies $M(p(t),0) = 0$, $0 \le t \le 1$. Also, Hypothesis (2.7) implies that $\dim \mathcal{N}(\Lambda(t)) = 1 = \operatorname{codim} \mathcal{R}(\Lambda(t))$, where $\Lambda(t) = \partial M(p(t),0)/\partial x$. Furthermore, the function $\dot{p}(t)$ is a basis for $\mathcal{N}(\Lambda(t))$ and a complement for $\mathcal{R}(\Lambda(t))$. It is now an obvious calculation to see that the functions $\alpha_1(t), \alpha_2(t)$ in (2.1) are given by

$$\alpha_1(t) = -\int_0^1 h(s)\dot{p}(s+t)^2 ds, \quad \alpha_2(t) = \int_0^1 \dot{p}(s+t)f(s)ds. \tag{2.8}$$

If (α_1,α_2) satisfy (H_1)-(H_3), then the above results are directly applicable to the determination of the bifurcation curves for the 1-periodic solutions of Equation (2.5) which lie in a neighborhood of the periodic orbit $\Gamma \subseteq \mathbb{R}^2$ of Equation (2.6) defined by $\Gamma = \{(p(s), dp(s)/ds), 0 \le s \le 1$. For $h(s) = 1$, $0 \le s \le 1$, these results were previously obtained in [5]. A detailed explanation of the manner in which the 1-periodic solutions wind onto the cylinder $\Gamma \times \mathbb{R}$ as $\lambda \to 0$ along a curve $\gamma \in \mathbb{R}^2$ is given in [5]. Also, a reasonable physical explanation for the discontinuities of the solutions at $\lambda = 0$ is given in [5].

§3. Proof of the results.

Our objective in this section is to give the essential elements of the proofs of the results of Section 2. The notation of that section will be used without explanation.

By the Implicit Function Theorem and the compactness of Γ, one obtains the following result.

Lemma 3.1. There is a $\delta > 0$ such that the transformation $x \mapsto (t,y)$,

$$x = p(t) + y, \quad y \in [I-U(t)]X, \tag{3.1}$$

from a neighborhood of Γ to $[0,1) \times (I-U(t))X$ is a diffeomorphism for $t \in [0,1)$, $|y| < \delta$.

For the determination of all solutions of Equation (1.1) in a sufficiently small neighborhood of Γ, Lemma 3.1 implies that it is sufficient to consider x given by Equation (3.1) and $|y|$ in a sufficiently small neighborhood of zero. If x is a solution of

Equation (1.1) and y is defined by Equation (3.1), then y satisfies the equation

$$0 = M(p(t) + y,\lambda) \stackrel{\text{def}}{=} A(t)y + N(t,y,\lambda) \tag{3.2}$$

where $N(t,y,\lambda) = M(p(t) + y,\lambda) - A(t)y$. By the boundary conditions on p, we may consider this equation for $t \in R$. Decomposing this equation into its components in $E(t)Z$ and $[I-E(t)]Z$ and using the fact that $A(t)$ as a mapping from $[I-U(t)]X$ onto $E(t)Z$ has a bounded inverse (this is the method of Liapunov-Schmidt), there exist $\lambda_0 > 0$, $\delta > 0$ and a unique function $y^* \in C^2(\mathbb{R} \times \{|\lambda| < \lambda_0\}, (I-U(t))X)$, $y^*(t,0) = 0$ for all t, such that Equation (3.2) has a solution for $t \in \mathbb{R}$, $|y| < \delta$, $|\lambda| < \lambda_0$ if and only if $y = y^*(t,\lambda)$ and (t,λ) satisfies the bifurcation equation

$$0 = F(t,\lambda) \stackrel{\text{def}}{=} (I-E(t))[M(p(t) + y^*(t,\lambda),\lambda) - A(t)y^*(t,\lambda)] \tag{3.3}$$

If we define the scalar function $f(t,\lambda)$ by the relation

$$F(t,\lambda) \stackrel{\text{def}}{=} f(t,\lambda)q(t) \tag{3.4}$$

then the bifurcation equation is equivalent to the scalar equation

$$f(t,\lambda) = 0 \tag{3.5}$$

for $t \in \mathbb{R}$, $|\lambda| < \lambda_0$. Since $y^*(t,0) = 0$ for all t, it follows that

$$f(t,0) = 0, \quad t \in \mathbb{R}. \tag{3.6}$$

Relation (3.6) reflects the fact that Equation (1.2) does not have an isolated solution. Equality (3.6) is the basic reason why this problem differs from the usual bifurcation problem.

If $\Lambda = R^2$, $\lambda = (\lambda_1,\lambda_2) \in \mathbb{R}^2$, then

$$\partial f(t,0)/\partial\lambda_j = \alpha_j(t), \quad j = 1,2 \tag{3.7}$$

where each α_j, $j = 1,2$, is defined in Equation (2.1). The function f can thus be written as

$$f(t,\lambda) = \alpha(t)\cdot\lambda + h(t,\lambda)$$

where $h(t,0) = 0$, $\partial h(t,0)/\partial\lambda = 0$. For any $\lambda \neq 0$, solving Equation (3.5) is equivalent to solving the equation

$$\alpha(t)\cdot(\lambda/|\lambda|) + h(t,\lambda)/|\lambda| = 0. \tag{3.8}$$

If $\lambda/|\lambda| = \beta$, $H(t,\beta,|\lambda|) = h(t,\beta|\lambda|)/|\lambda|$, then $\beta \in S^1 = \{\beta \in \mathbb{R}^2: |\beta| = 1\}$ and H is C^2 in its arguments. The discussion of the solutions of Equation (3.8) becomes equivalent to the discussion of the equation

$$G(t,\beta,s) \stackrel{\text{def}}{=} \alpha(t)\cdot\beta + H(t,\beta,s) = 0 \tag{3.9}$$

for $t \in [0,1)$, $\beta \in S^1$, s small and nonnegative. In the following, we always understand $\beta \in S^1$ even though it may not be said explicitly.

If $\alpha(t_0)\cdot\beta_0 = 0$, then $G(t_0,\beta_0,0) = 0$. If $\alpha'(t_0)\cdot\beta_0 \neq 0$, then $\partial G(t_0,\beta_0,0)/\partial t \neq 0$ and the Implicit Function Theorem implies there is an $s_0 = s_0(t_0,\beta_0) > 0$ and a unique solution $t^*(\beta,s)$ of Equation (3.9) for $|\beta-\beta_0| < s_0$, $0 \leq s < s_0$, $t^*(\beta_0,0) = t_0$.

To complete the proof, we need to reformulate Hypotheses (H_2), (H_3) in an equivalent form. The vector $\alpha(t)$ defines a continuous linear functional on $\mathbb{R}^2$ by the relation $\alpha(t)\cdot\lambda$, $\lambda \in \mathbb{R}^2$. If the null space of $\alpha(t)$ is denoted by $\mathfrak{N}(\alpha(t))$, then $(\alpha_2(t),-\alpha_1(t))$ is a basis for $\mathfrak{N}(\alpha(t))$. By computing ϕ',ϕ'', one easily observes that (H_2), (H_3) are equivalent to

(H_2) The vector $\alpha'(t) \in \mathbb{R}^2$ is orthogonal to $\mathfrak{N}(\alpha(t))$ at most at a finite number of points $\{t^k,\ k = 1,2,\ldots,n\} \subseteq [0,1)$ and $\alpha''(t^k)$ is not orthogonal to $\mathfrak{N}(\alpha(t^k))$ for any $k = 1,2,\ldots,n$.

(H_3) The lines through the origin and $\alpha(t^j)$ and $\alpha(t^k)$ are not colinear for $j \neq k$, $j,k = 1,2,\ldots,n$.

The numbers t^k here are the same as before.

If $\alpha'(t_0)\cdot\beta_0 = 0$, then Hypothesis (H_2) implies $\alpha''(t_0)\cdot\beta_0 \neq 0$. Thus, the Implicit Function Theorem implies there is an $s_0 = s_0(t_0,\beta_0) > 0$ and a function $t^*(\beta,s)$, $t^*(\beta_0,0) = t_0$, such that

$$\partial G(t^*(\beta,s),\beta,s)/\partial t = 0$$

for $|\beta-\beta_0| < s_0$, $0 \leq s < s_0$ and $t^*(\beta,s)$ is unique in the region $|t-t_0| < s_0$. Thus, the function $\Omega(\beta,s) \stackrel{\text{def}}{=} G(t^*(\beta,s),\beta,s)$ is a maximum or minimum of $G(t,\beta,s)$ with respect to t at (β,s). A few elementary calculation show that $\Omega(\beta_0,0) = 0$, $\partial\Omega(\beta_0,0)/\partial\beta = \alpha(t_0)$. Therefore, the derivative of $\Omega(\beta,s)$ with respect to β on the sphere S^1 at $\beta = \beta_0$, $s = 0$ is the inner product of the vector $\alpha(t_0)$ with a unit vector orthogonal to β_0. But this vector will be a nonzero constant times $\alpha(t_0)$ since $\alpha(t_0) \neq 0$ by Hypothesis (H_1). Consequently, $\Omega(\beta_0,0)/\partial\beta$ on S^1 is nonzero. The Implicit Function Theorem implies there is a $\delta(\beta_0) > 0$ and a function $\beta^*(s) \in S^1$, $\beta^*(0) = \beta_0$, such that $\Omega(\beta^*(s),s) = 0$ for $0 \leq s < \delta(\beta_0)$ describes a curve. On one side of this curve, there are two simple solutions of Equation (3.9) and no solutions on the other side. In terms of the original coordinates (λ_1,λ_2), this implies there are two solutions of Equation (3.5) near t_0 on one side of the curve $\lambda = s\beta^*(s)$, $0 \leq s < \delta(\beta_0)$ and none on the other. This curve in λ-space is a bifurcation curve and is tangent to the line $\alpha(t_0)\cdot\ = 0$ at $\lambda = 0$. The fact that the number of solutions increases or decreases as stated in the theorem is clear.

The above analysis can be applied to each of the points t^j in Hypothesis (H_2) to obtain an $s_0 > 0$ such that all solutions of Equation (3.9) for $|t-t^j| < s_0$, $\beta \in S^1$, $0 \leq s < s_0$ are determined by the argument above. We obtain the bifurcation curves as well. The complement of the intervals $|t-t^j| < s_0$, $j = 1,2,\ldots,n$, in $[0,1)$ is compact and $\alpha'(t_0)\cdot\beta_0 \neq 0$ for any t_0,β_0 satisfying $\alpha(t_0)\cdot\beta_0 = 0$. A repeated application of the Implicit Function Theorem shows one can choose s_0 so that no further bifurcations occur in this complement for any $\beta \in S^1$, $0 \leq s < s_0$. Returning to the original coordinates (λ_1,λ_2), we see that the complete bifurcation diagram has been obtained for a full neighborhood of $\lambda = 0$.

To describe precisely the bifurcation pattern as stated in Theorem 2.1, we need to know that no two bifurcation curves obtained by the above process coincide. This is the only reason Hypothesis (H_3) is imposed. This proves the first part of Theorem 2.1.

The last part of the theorem is clear from the definitions of the terms involved. This completes the proof of Theorem 2.1.

To prove Theorem 2.2, we first note that the method of Liapunov-Schmidt implies there is a $\tau_0 > 0$ and a continuous function $t(\tau) \subseteq [0,1]$, $0 < \tau < \tau_0$ such that the solution $x(\tau)$ is given by

$$x(\tau) = p(t(\tau)) + y^*(t(\tau),\lambda(\tau)), \quad 0 < \tau < \tau_0,$$

where $y^*(t,\lambda)$ is the solution of Equation (3.2), $(t(\tau),\lambda(\tau))$ satisfy Equation (3.5) or, equivalently, Equation (3.8). Suppose there is a sequence $\tau_j \to 0$ such that $\cot^{-1}(\lambda_1(\tau_j)/\lambda_2(\tau_j)) \to \phi_0 \in [0,2\pi]$ as $j \to \infty$, or, equivalently, $\lambda(\tau_j)/|\lambda(\tau_j)| \to \beta_0 \in S_1$ as $j \to \infty$. Without loss of generality, we may assume $t(\tau_j) \to t_0 \in [0,1]$ as $j \to \infty$. Then $x(\tau_j) \to p(t_0)$, $\alpha(t_0)\cdot\beta_0 = 0$, $\phi(t_0) = \phi_0$. Since all functions are continuous, the conclusion of Theorem 2.2 follows immediately, and the proof is complete.

To prove Corollary 2.1, suppose the conditions of Theorem 2.2 are satisfied and the interval $[\phi_m(\gamma),\phi_M(\gamma)]$ consists of more than one point, then $x(\tau)$ cannot have a limit as $\tau \to 0$ although every limit point satisfies Equation (1.2). If $x(\tau)$ has a limit as $\tau \to 0$, then $\mathcal{S}(\gamma)$ is precompact and it is, therefore, necessary that $\cot^{-1}(\lambda_1(\tau)/\tau_2(\tau))$ approach a limit as $\tau \to 0$. Conversely, if $\cot^{-1}(\lambda_1(\tau)/\lambda_2(\tau)) \to \phi_0$ as $\tau \to 0$, then $\lambda(\tau)/|\lambda(\tau)| \to \beta_0 \in S^1$ as $\tau \to 0$ and $\alpha(t(\tau))\cdot\beta_0 \to 0$ as $\tau \to 0$ from the argument used in the proof of Theorem 2.2. Hypothesis (H_2) implies the set of $t \in [0,1]$ such that $\alpha(t)\cdot\beta_0 = 0$ is isolated. Since $t(\tau)$ is continuous for $0 < \tau \leq 1$, this implies $t(\tau) \to t_0 \in \{t^k, k = 1,2,\ldots,n\}$ as $\tau \to 0$. The argument used in the proof of Theorem 2.2 implies $x(\tau) \to p(t_0)$ as $t \to \infty$ and the proof of Corollary 2.1 is complete.

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This research was supported in part by the Air Force Office of Scientific Research under AFOSR 71-2078D, in part by the National Science Foundation under MPS 71-02923 and in part by the U.S. Army Research Office under AROD AAG 29-76-6-0052.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

18

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 19 AFOSR - TR - 77 - 0571	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 9
4. TITLE (and Subtitle) 6 BIFURCATION NEAR FAMILIES OF SOLUTIONS		5. TYPE OF REPORT & PERIOD COVERED Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. K. Hale 10 Jack K. Hale		8. CONTRACT OR GRANT NUMBER(s) AFOSR 71-2078
9. PERFORMING ORGANIZATION NAME AND ADDRESS Brown University Lefschetz Center for Dynamical Systems Providence, RI 02912		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB DC 20332		12. REPORT DATE 11 Mar 77
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 15p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimites.

15 AF-AFOSR-2078-71,

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NSF-MPS-71-02923

18. SUPPLEMENTARY NOTES

16 2304 17 A1

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Many investigations in bifurcation theory are concerned with the following problem. If $M(0,0) = 0$ and $\partial M(0,0)/\partial x$ has a non-trivial null space, find all solutions of the equation

$$M(x,\lambda) = 0 \qquad (1.1)$$

for (x,λ) in a neighborhood of $(0,0) \in X \times \Lambda$.

If $\dim \Lambda = 1$; that is, there is only one parameter involved

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then the existence of more than one solution in a neighborhood of zero can be proved by making assumptions only about $M(0,0)/\partial x$ and $M(0,0)/\partial x\partial\lambda$. However, if $\dim \Lambda \geq 2$, then the problem is much more difficult and more detailed information is needed about the function M. A careful examination of the existing literature for $\dim \Lambda \geq 2$ reveals that the additional conditions imposed on M imply, in particular, that the solution $x = 0$ of the equation

$$M(x,0) = 0 \tag{1.2}$$

is isolated (see, for example, the papers on catastrophe theory). These hypotheses eliminate the possibility that Equation (1.2) has a family of solutions containing $x = 0$. Such a situation occurs, for example, for $M(x,\lambda) = Ax + N(x,\lambda)$, where A is linear with a nontrivial null space and $N(x,0) = 0$ for all x. There also are interesting applications where Equation (1.2) is nonlinear and there exists a family of solutions. For example, Equation (1.2) could be an autonomous ordinary differential equation with a nonconstant periodic orbit of period 2Π with the family of solutions being obtained by a phase shift. When the differential equation in the latter situation is a Hamiltonian system, the parameters (λ_1,λ_2) could correspond to a small damping term and a small forcing term of period 2Π. To the authors knowledge, the first complete investigations of special problems of each of these latter types are contained in papers by Hale, Táboas and Rodrigues.

It is the purpose of this paper to begin the investigation of the abstract problem for Equation (1.1), especially to extend the results in the paper by Hale and Táboas.